Communication Systems

Lecture 7

Some Properties of Fourier Transform

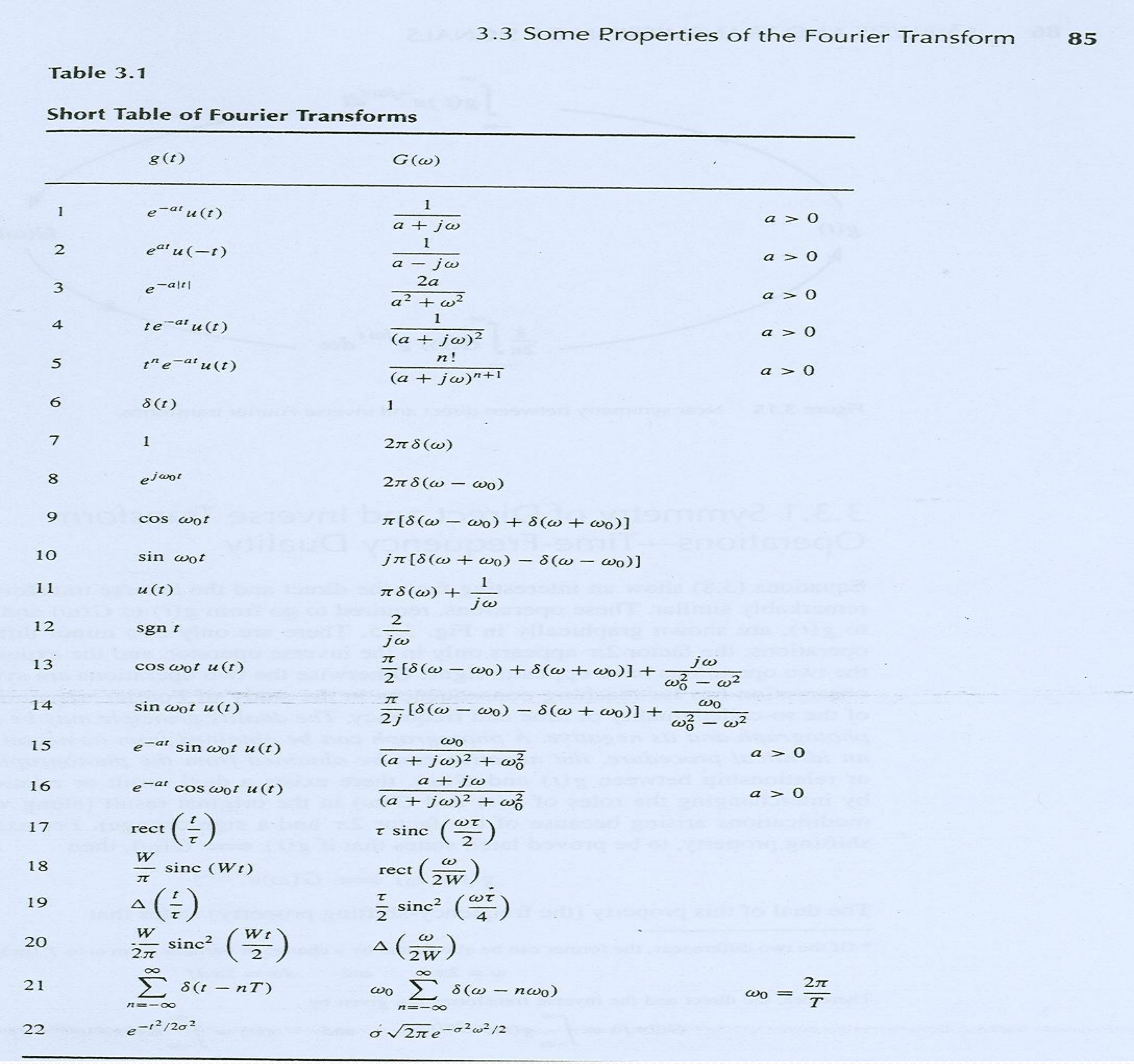
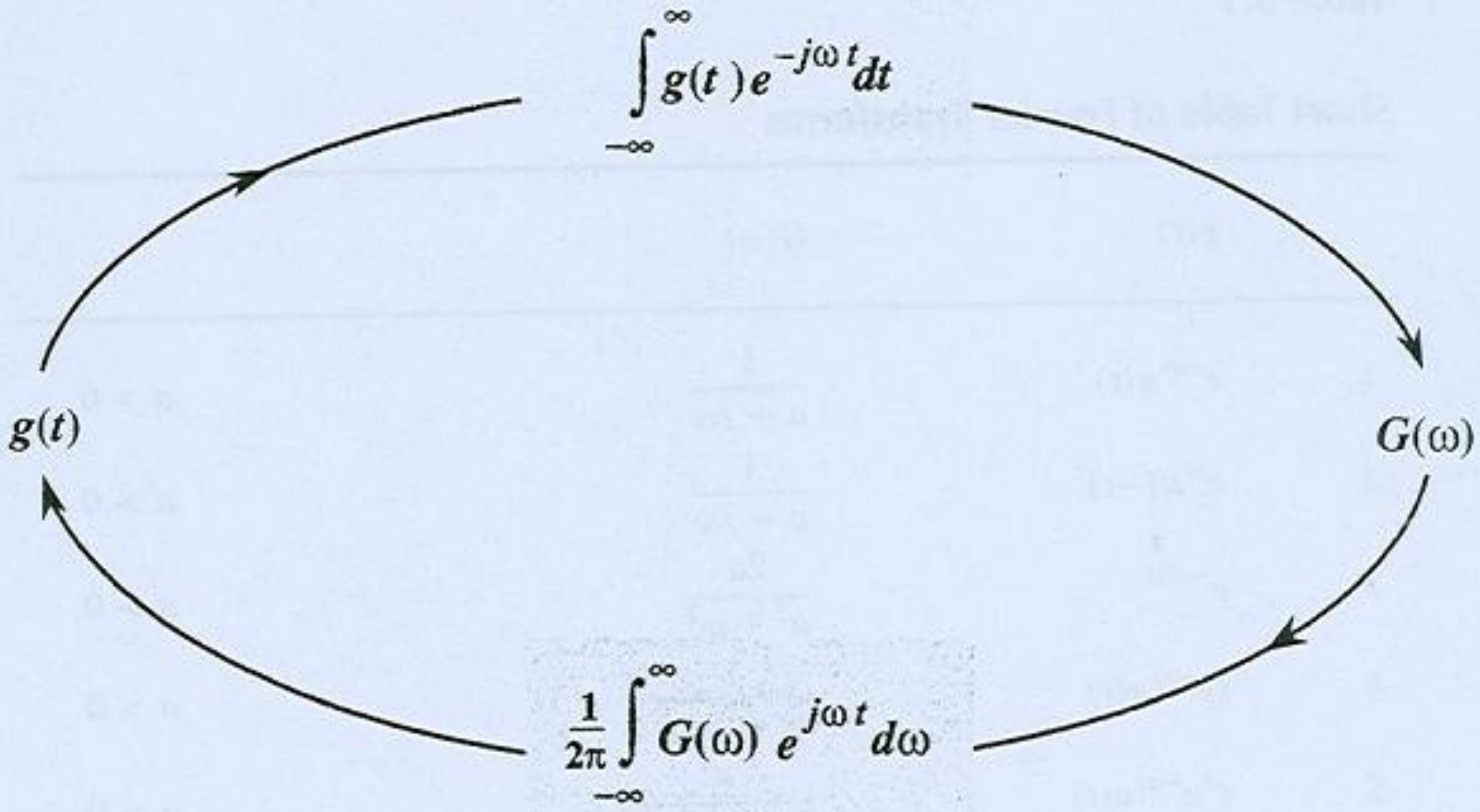


Table 3.1

Short Table of Fourier Transforms

	$g(t)$	$G(\omega)$	
1	$e^{-at}u(t)$	$\frac{1}{a+j\omega}$	$a > 0$
2	$e^{at}u(-t)$	$\frac{1}{a-j\omega}$	$a > 0$
3	$e^{-a\|t\|}$	$\frac{2a}{a^2+\omega^2}$	$a > 0$
4	$te^{-at}u(t)$	$\frac{1}{(a+j\omega)^2}$	$a > 0$
5	$t^n e^{-at}u(t)$	$\frac{n!}{(a+j\omega)^{n+1}}$	$a > 0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	
9	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	
10	$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	
11	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$	
12	$\operatorname{sgn} t$	$\frac{2}{j\omega}$	
13	$\cos\omega_0 t\, u(t)$	$\frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\frac{j\omega}{\omega_0^2-\omega^2}$	
14	$\sin\omega_0 t\, u(t)$	$\frac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\frac{\omega_0}{\omega_0^2-\omega^2}$	
15	$e^{-at}\sin\omega_0 t\, u(t)$	$\frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$	$a > 0$
16	$e^{-at}\cos\omega_0 t\, u(t)$	$\frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$a > 0$
17	$\operatorname{rect}\left(\frac{t}{\tau}\right)$	$\tau\operatorname{sinc}\left(\frac{\omega\tau}{2}\right)$	
18	$\frac{W}{\pi}\operatorname{sinc}(Wt)$	$\operatorname{rect}\left(\frac{\omega}{2W}\right)$	
19	$\Delta\left(\frac{t}{\tau}\right)$	$\frac{\tau}{2}\operatorname{sinc}^2\left(\frac{\omega\tau}{4}\right)$	
20	$\frac{W}{2\pi}\operatorname{sinc}^2\left(\frac{Wt}{2}\right)$	$\Delta\left(\frac{\omega}{2W}\right)$	
21	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$	$\omega_0=\frac{2\pi}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$	



$$g(t)$$

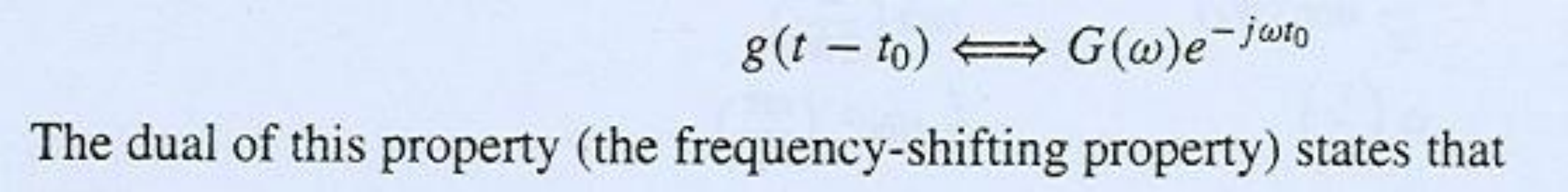
$$\int_{-\infty}^{\infty} g(t)\, e^{-j\omega t} dt$$

$$G(\omega)$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega)\; e^{j\omega t} d\omega$$

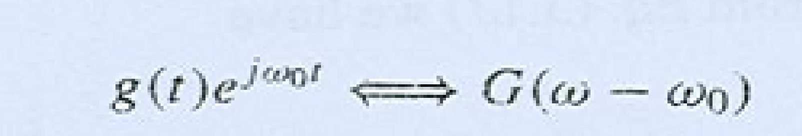
Figure 3.15 Near symmetry between direct and inverse Fourier transforms.

Symmetry of Direct and Inverse Transform Operations – Time Frequency Duality



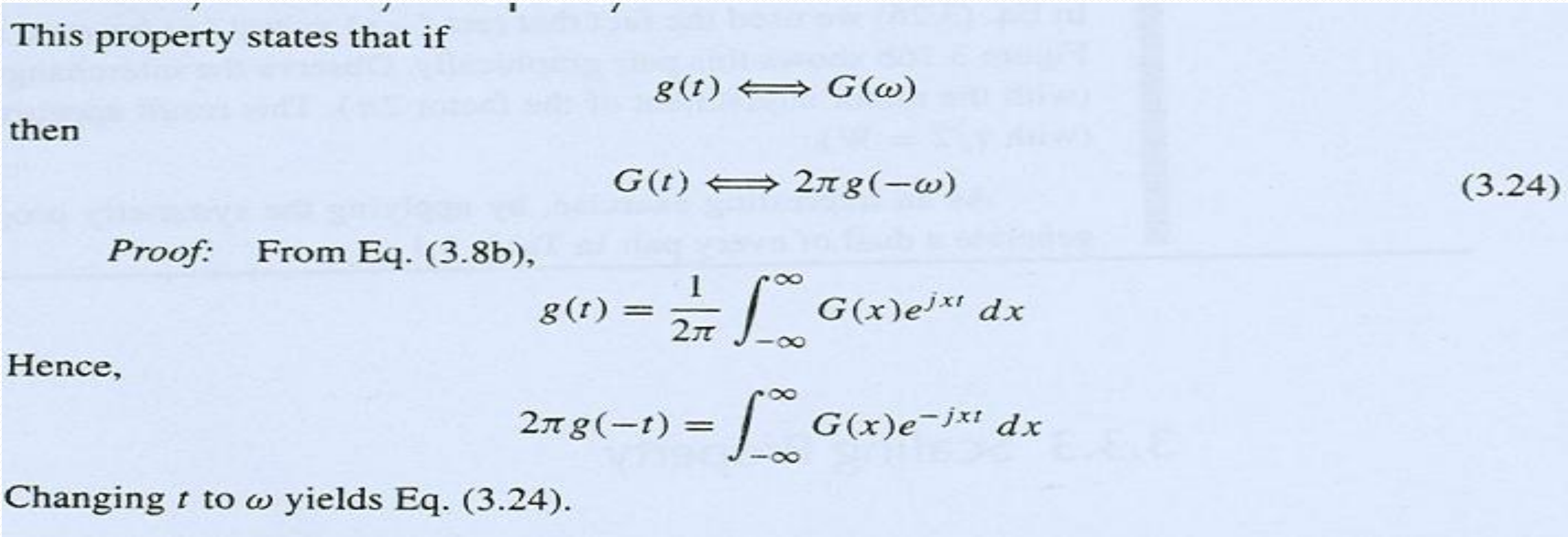
$$g(t-t_0) \Longleftrightarrow G(\omega)e^{-j\omega t_0}$$

The dual of this property (the frequency-shifting property) states that



$$g(t)e^{j\omega_0 t} \Longleftrightarrow G(\omega-\omega_0)$$

Symmetry Property



This property states that if

$$g(t) \Longleftrightarrow G(\omega)$$

then

$$G(t) \Longleftrightarrow 2\pi g(-\omega) \tag{3.24}$$

Proof: From Eq. (3.8b),

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(x)e^{jxt}\,dx$$

Hence,

$$2\pi g(-t) = \int_{-\infty}^{\infty} G(x)e^{-jxt}\,dx$$

Changing t to ω yields Eq. (3.24).

Example 3.8 In this example we shall apply the symmetry property [Eq, (3.24)] to the pair in Fig 3.16a.

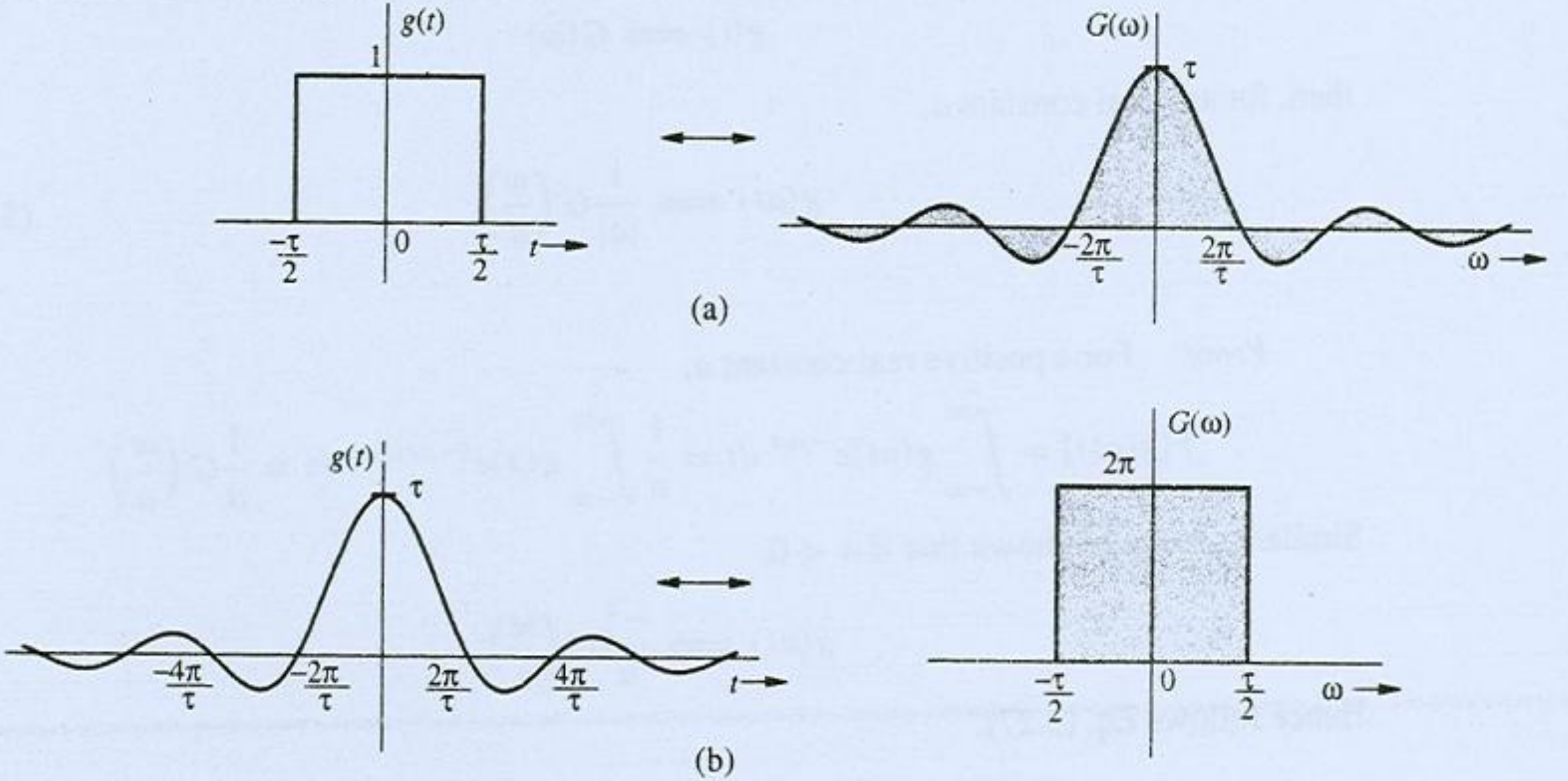
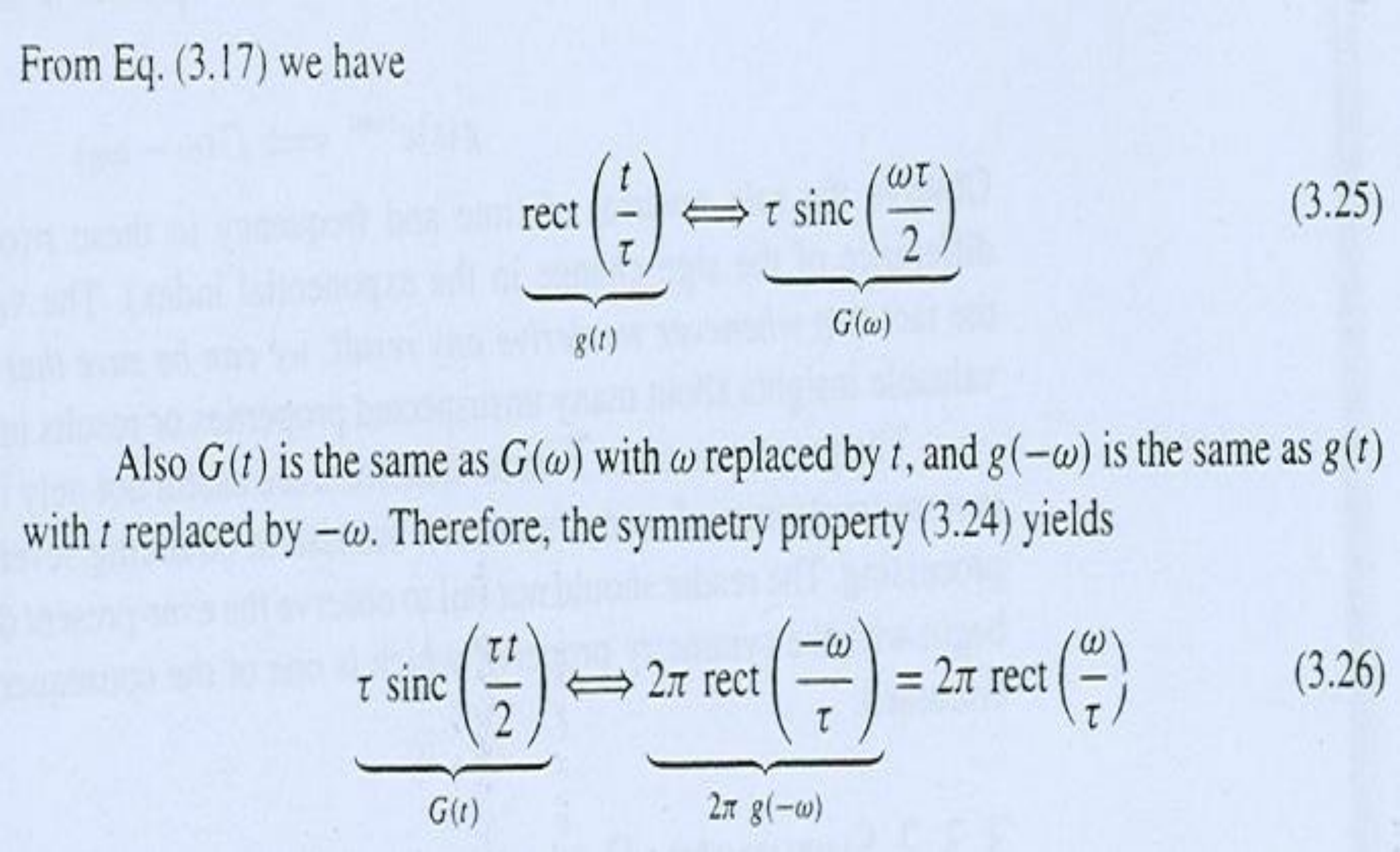


Figure 3.16 Duality property of the Fourier transform.



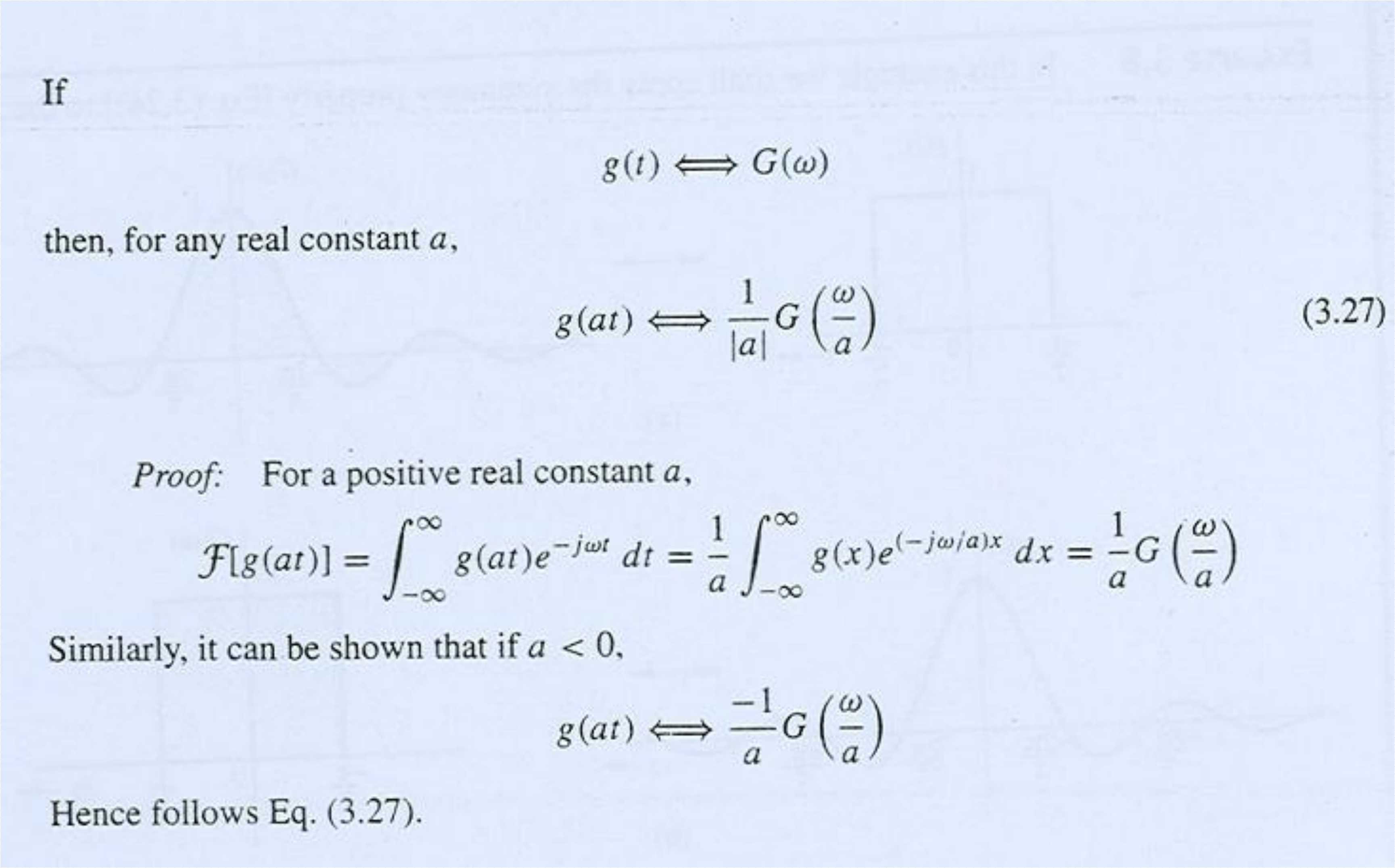
From Eq. (3.17) we have

$$\underbrace{\operatorname{rect}\left(\frac{t}{\tau}\right)}_{g(t)} \Longleftrightarrow \underbrace{\tau \operatorname{sinc}\left(\frac{\omega\tau}{2}\right)}_{G(\omega)} \tag{3.25}$$

Also $G(t)$ is the same as $G(\omega)$ with ω replaced by t, and $g(-\omega)$ is the same as $g(t)$ with t replaced by $-\omega$. Therefore, the symmetry property (3.24) yields

$$\underbrace{\tau \operatorname{sinc}\left(\frac{\tau t}{2}\right)}_{G(t)} \Longleftrightarrow \underbrace{2\pi \operatorname{rect}\left(\frac{-\omega}{\tau}\right)}_{2\pi\ g(-\omega)} = 2\pi \operatorname{rect}\left(\frac{\omega}{\tau}\right) \tag{3.26}$$

Scaling Property



If

$$g(t) \Longleftrightarrow G(\omega)$$

then, for any real constant a,

$$g(at) \Longleftrightarrow \frac{1}{|a|} G\left(\frac{\omega}{a}\right) \tag{3.27}$$

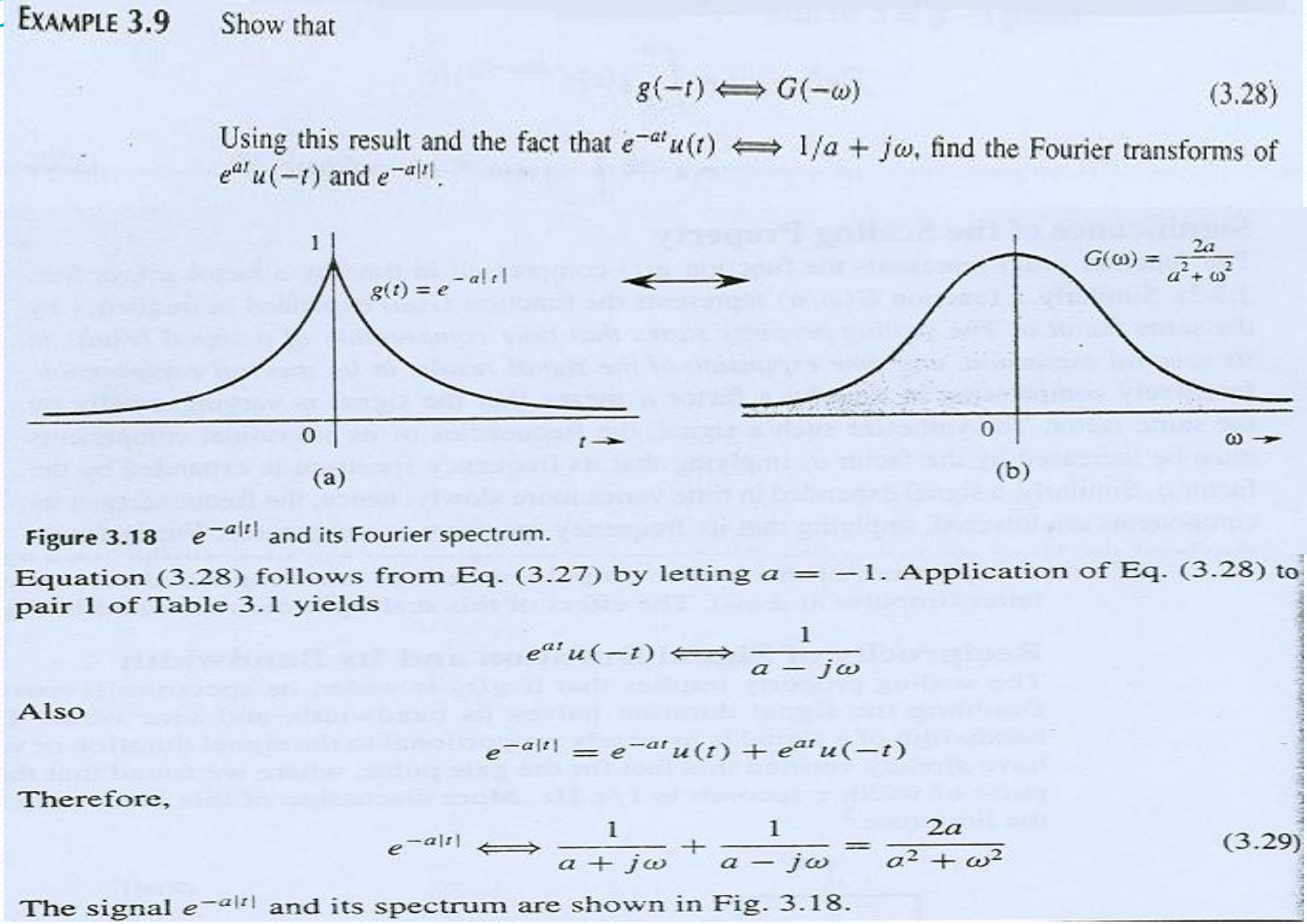
Proof: For a positive real constant a,

$$\mathcal{F}[g(at)] = \int_{-\infty}^{\infty} g(at) e^{-j\omega t}\, dt = \frac{1}{a} \int_{-\infty}^{\infty} g(x) e^{(-j\omega/a)x}\, dx = \frac{1}{a} G\left(\frac{\omega}{a}\right)$$

Similarly, it can be shown that if $a < 0$,

$$g(at) \Longleftrightarrow \frac{-1}{a} G\left(\frac{\omega}{a}\right)$$

Hence follows Eq. (3.27).



EXAMPLE 3.9 Show that

$$g(-t) \Longleftrightarrow G(-\omega) \tag{3.28}$$

Using this result and the fact that $e^{-at}u(t) \Longleftrightarrow 1/a + j\omega$, find the Fourier transforms of $e^{at}u(-t)$ and $e^{-a|t|}$.

Figure 3.18 $e^{-a|t|}$ and its Fourier spectrum.

Equation (3.28) follows from Eq. (3.27) by letting $a = -1$. Application of Eq. (3.28) to pair 1 of Table 3.1 yields

$$e^{at}u(-t) \Longleftrightarrow \frac{1}{a - j\omega}$$

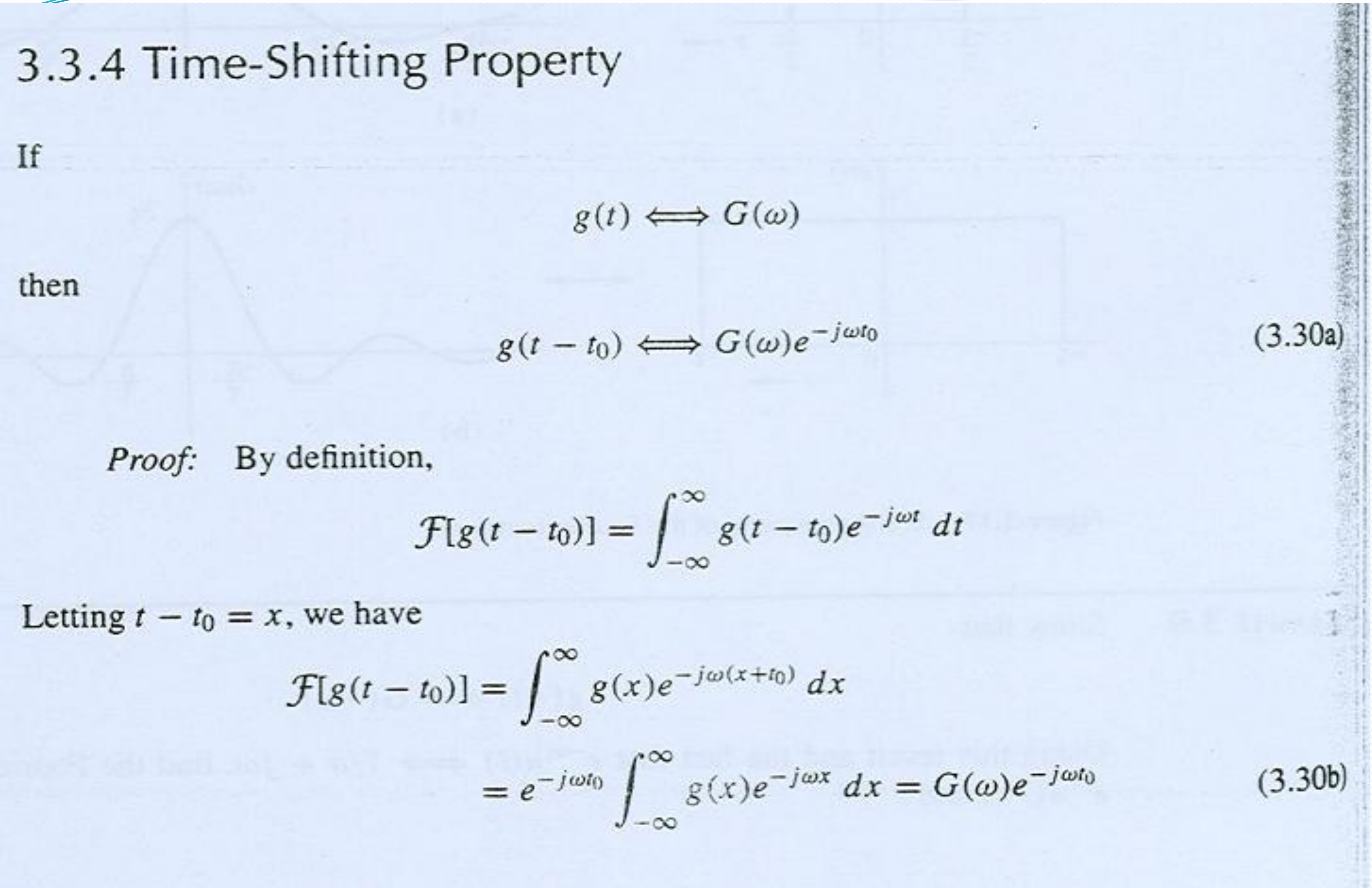
Also

$$e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t)$$

Therefore,

$$e^{-a|t|} \Longleftrightarrow \frac{1}{a + j\omega} + \frac{1}{a - j\omega} = \frac{2a}{a^2 + \omega^2} \tag{3.29}$$

The signal $e^{-a|t|}$ and its spectrum are shown in Fig. 3.18.



3.3.4 Time-Shifting Property

If

$$g(t) \Longleftrightarrow G(\omega)$$

then

$$g(t - t_0) \Longleftrightarrow G(\omega)e^{-j\omega t_0} \tag{3.30a}$$

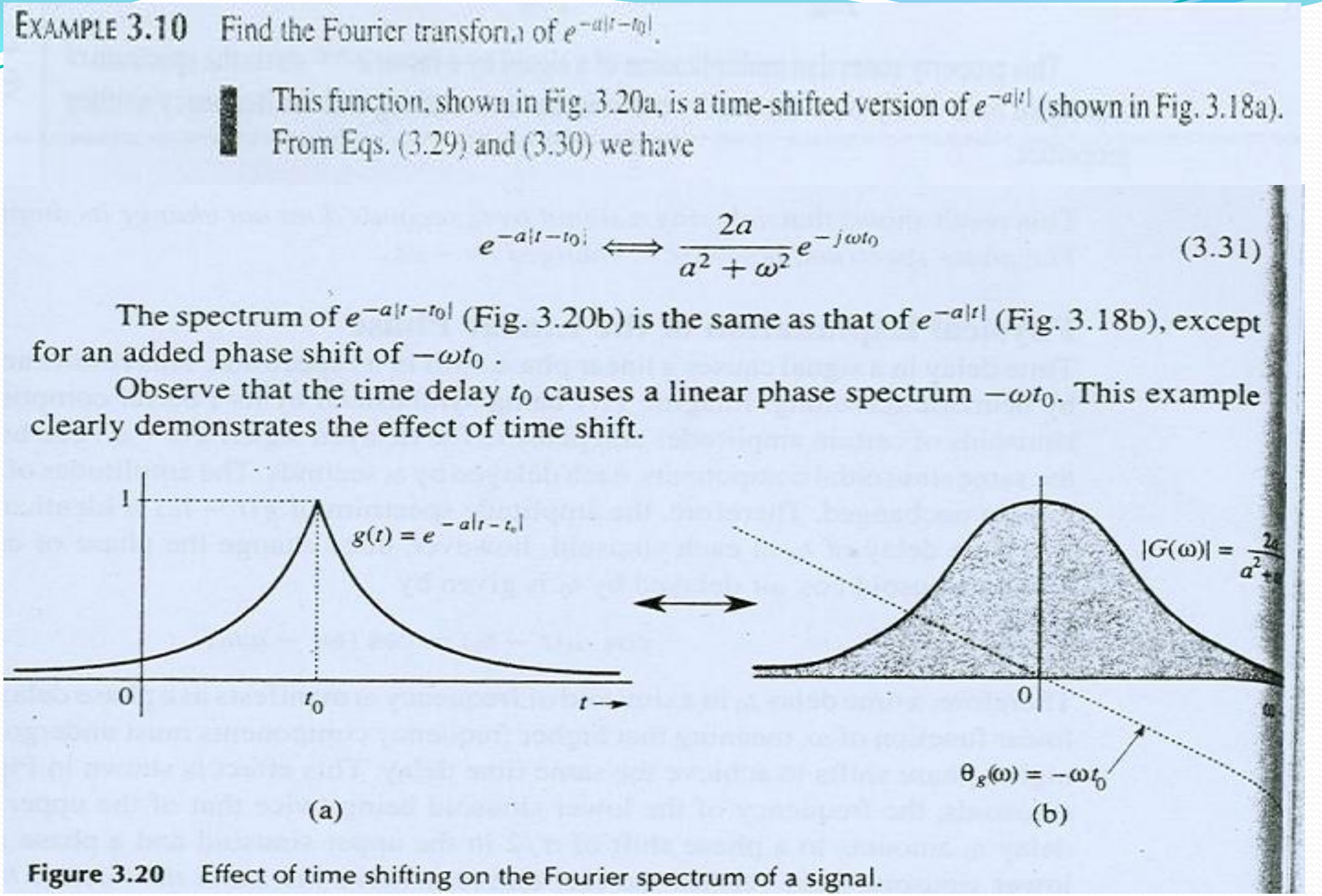
Proof: By definition,

$$\mathcal{F}[g(t - t_0)] = \int_{-\infty}^{\infty} g(t - t_0)e^{-j\omega t}\, dt$$

Letting $t - t_0 = x$, we have

$$\mathcal{F}[g(t - t_0)] = \int_{-\infty}^{\infty} g(x)e^{-j\omega(x+t_0)}\, dx$$

$$= e^{-j\omega t_0}\int_{-\infty}^{\infty} g(x)e^{-j\omega x}\, dx = G(\omega)e^{-j\omega t_0} \tag{3.30b}$$



EXAMPLE 3.10 Find the Fourier transform of $e^{-a|t-t_0|}$.

This function, shown in Fig. 3.20a, is a time-shifted version of $e^{-a|t|}$ (shown in Fig. 3.18a). From Eqs. (3.29) and (3.30) we have

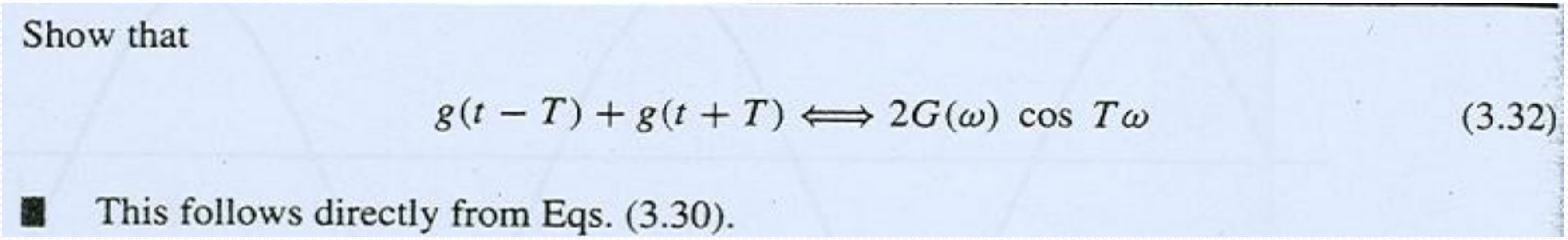
$$e^{-a|t-t_0|} \Longleftrightarrow \frac{2a}{a^2+\omega^2}e^{-j\omega t_0} \tag{3.31}$$

The spectrum of $e^{-a|t-t_0|}$ (Fig. 3.20b) is the same as that of $e^{-a|t|}$ (Fig. 3.18b), except for an added phase shift of $-\omega t_0$.

Observe that the time delay t_0 causes a linear phase spectrum $-\omega t_0$. This example clearly demonstrates the effect of time shift.

Figure 3.20 Effect of time shifting on the Fourier spectrum of a signal.

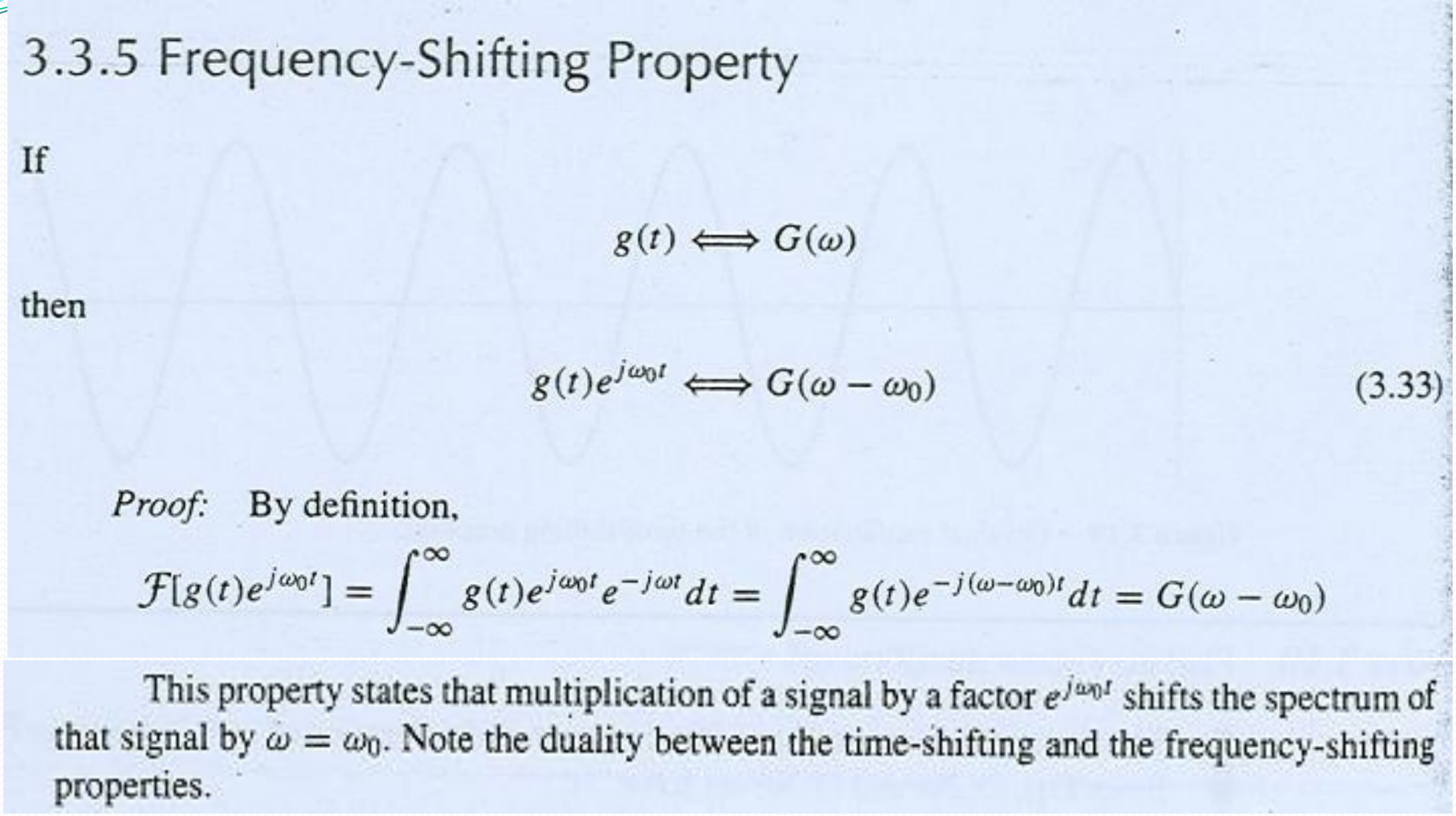
Example 3.11



Show that

$$g(t-T)+g(t+T) \Longleftrightarrow 2G(\omega)\ \cos\ T\omega \tag{3.32}$$

■ This follows directly from Eqs. (3.30).



3.3.5 Frequency-Shifting Property

If

$$g(t) \Longleftrightarrow G(\omega)$$

then

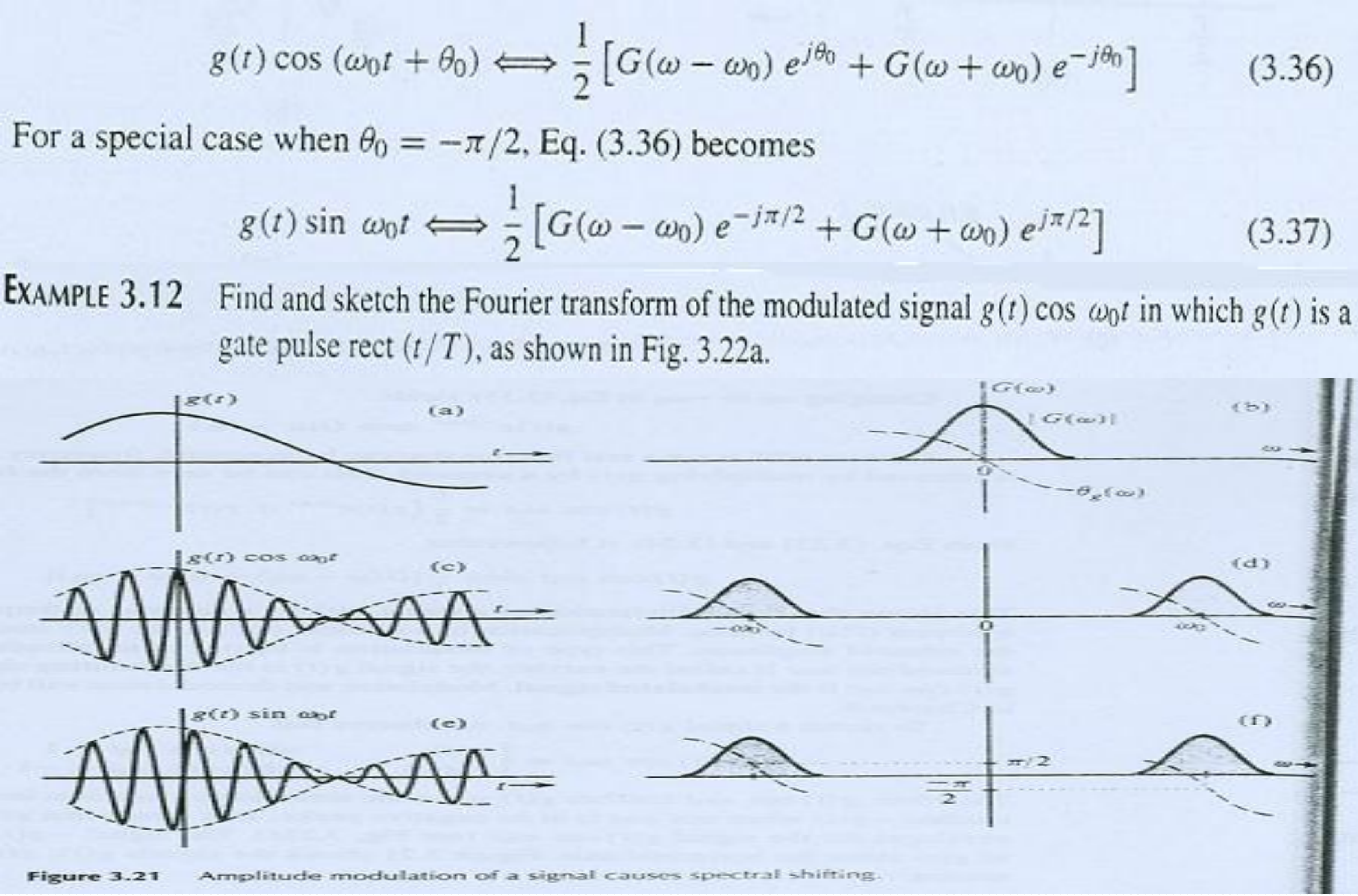
$$g(t)e^{j\omega_0 t} \Longleftrightarrow G(\omega - \omega_0) \tag{3.33}$$

Proof: By definition,

$$\mathcal{F}[g(t)e^{j\omega_0 t}] = \int_{-\infty}^{\infty} g(t)e^{j\omega_0 t}e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} g(t)e^{-j(\omega-\omega_0)t}\,dt = G(\omega - \omega_0)$$

This property states that multiplication of a signal by a factor $e^{j\omega_0 t}$ shifts the spectrum of that signal by $\omega = \omega_0$. Note the duality between the time-shifting and the frequency-shifting properties.

Shifting the Phase Spectrum of Modulated Signal



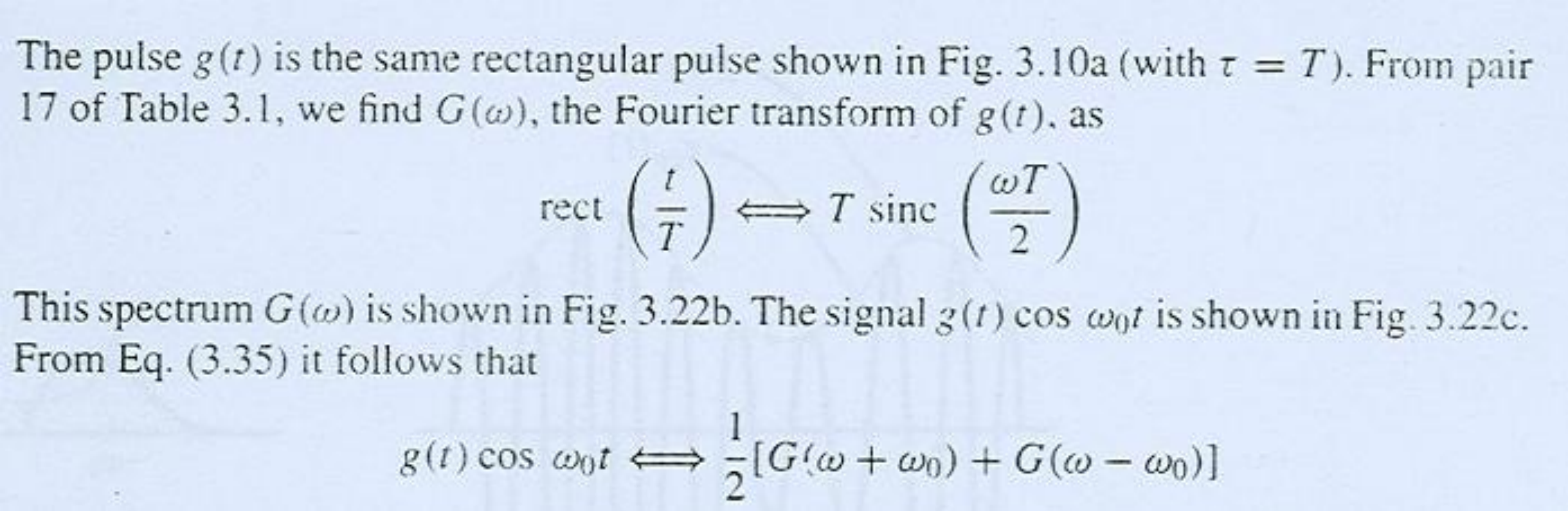
$$g(t)\cos(\omega_0 t + \theta_0) \Longleftrightarrow \frac{1}{2}\left[G(\omega - \omega_0)\, e^{j\theta_0} + G(\omega + \omega_0)\, e^{-j\theta_0}\right] \tag{3.36}$$

For a special case when $\theta_0 = -\pi/2$, Eq. (3.36) becomes

$$g(t)\sin\omega_0 t \Longleftrightarrow \frac{1}{2}\left[G(\omega - \omega_0)\, e^{-j\pi/2} + G(\omega + \omega_0)\, e^{j\pi/2}\right] \tag{3.37}$$

EXAMPLE 3.12 Find and sketch the Fourier transform of the modulated signal $g(t)\cos\omega_0 t$ in which $g(t)$ is a gate pulse rect (t/T), as shown in Fig. 3.22a.

Figure 3.21 Amplitude modulation of a signal causes spectral shifting.



The pulse $g(t)$ is the same rectangular pulse shown in Fig. 3.10a (with $\tau = T$). From pair 17 of Table 3.1, we find $G(\omega)$, the Fourier transform of $g(t)$, as

$$\operatorname{rect}\left(\frac{t}{T}\right) \Longleftrightarrow T \operatorname{sinc}\left(\frac{\omega T}{2}\right)$$

This spectrum $G(\omega)$ is shown in Fig. 3.22b. The signal $g(t)\cos\omega_0 t$ is shown in Fig. 3.22c. From Eq. (3.35) it follows that

$$g(t)\cos\omega_0 t \Longleftrightarrow \frac{1}{2}[G(\omega+\omega_0)+G(\omega-\omega_0)]$$

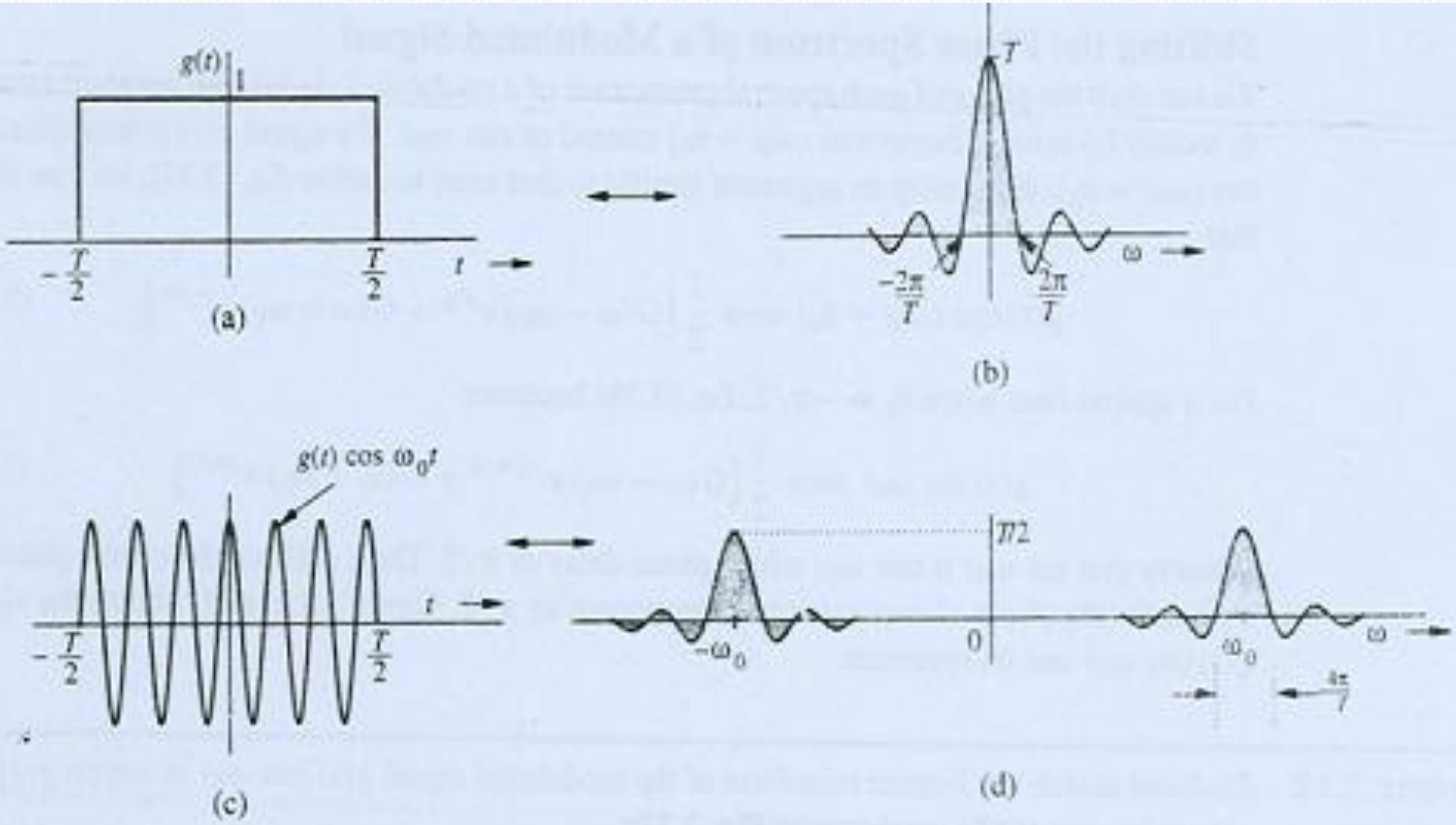


Figure 3.22 Example of spectral shifting by amplitude modulation.

EXAMPLE 3.13 Find the Fourier transform of a general periodic signal $g(t)$ of period T_0, and hence, determine the Fourier transform of the periodic impulse train $\delta_{T_0}(t)$ shown in Fig. 3.24a.

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* It is necessary that $2\pi B << \omega_0$ for a well-defined envelope. Otherwise the variations of $E(t)$ are of the same order as the carrier, and it will be difficult to separate the envelope from the carrier.

Figure 3.24 Impulse train and its spectrum.

A periodic signal $g(t)$ can be expressed as an exponential Fourier series as

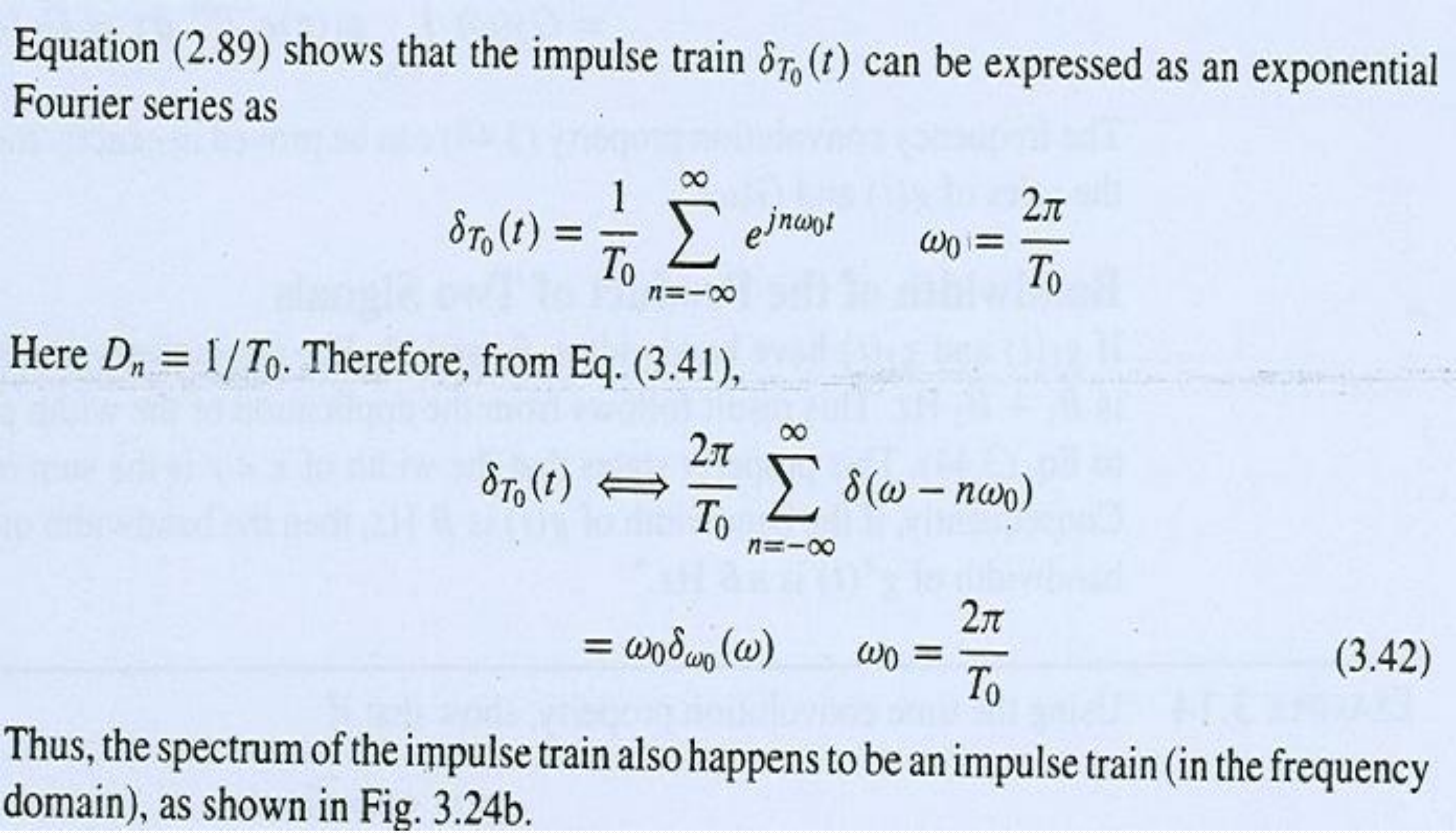
$$g(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

Therefore,

$$g(t) \Longleftrightarrow \sum_{n=-\infty}^{\infty} \mathcal{F}[D_n e^{jn\omega_0 t}]$$

Now from Eq. (3.20a), it follows that

$$g(t) \Longleftrightarrow 2\pi \sum_{n=-\infty}^{\infty} D_n \delta(\omega - n\omega_0) \qquad (3.41)$$



Equation (2.89) shows that the impulse train $\delta_{T_0}(t)$ can be expressed as an exponential Fourier series as

$$\delta_{T_0}(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

Here $D_n = 1/T_0$. Therefore, from Eq. (3.41),

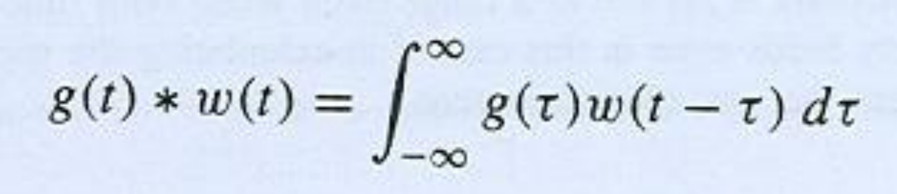
$$\delta_{T_0}(t) \Longleftrightarrow \frac{2\pi}{T_0} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$

$$= \omega_0 \delta_{\omega_0}(\omega) \qquad \omega_0 = \frac{2\pi}{T_0} \tag{3.42}$$

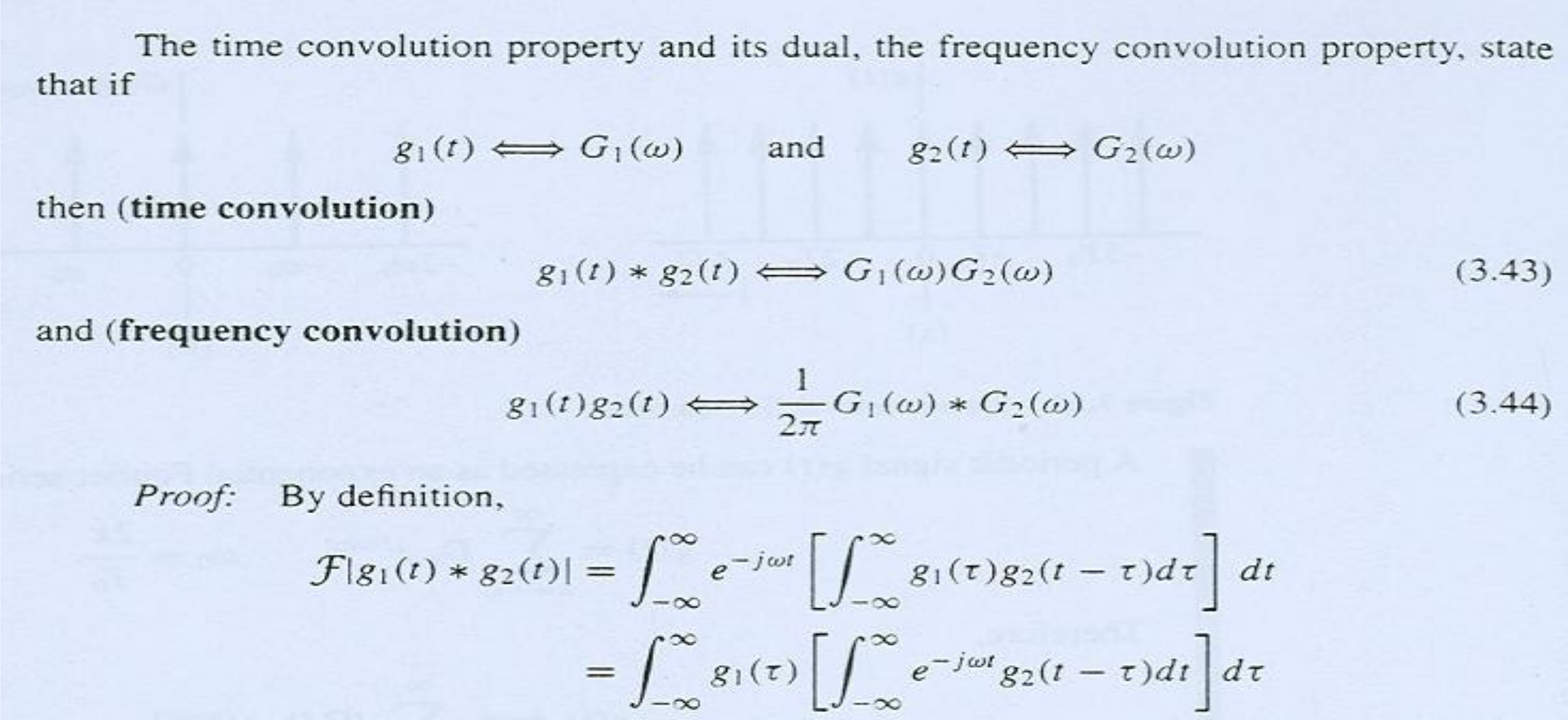
Thus, the spectrum of the impulse train also happens to be an impulse train (in the frequency domain), as shown in Fig. 3.24b.

Convolution

The convolution of two function *g(t)* and *w (t)*, denoted by *g(t) * w(t)*, is defined by the integral



$$g(t) * w(t) = \int_{-\infty}^{\infty} g(\tau) w(t-\tau)\, d\tau$$



The time convolution property and its dual, the frequency convolution property, state that if

$$g_1(t) \Longleftrightarrow G_1(\omega) \qquad \text{and} \qquad g_2(t) \Longleftrightarrow G_2(\omega)$$

then (**time convolution**)

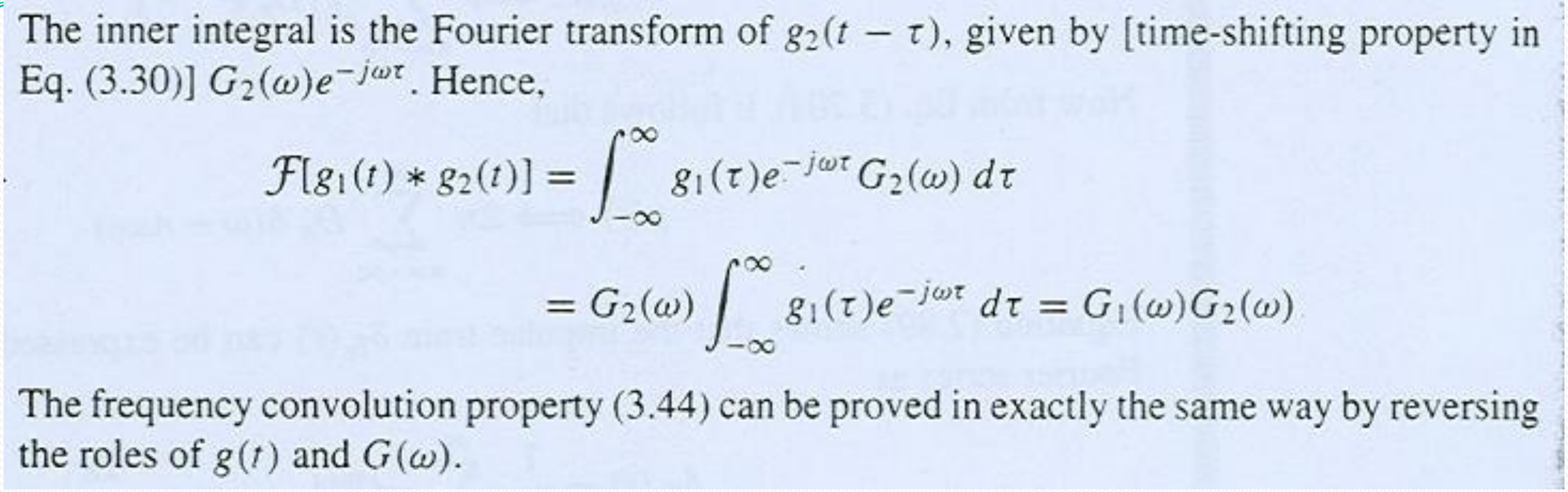
$$g_1(t) * g_2(t) \Longleftrightarrow G_1(\omega) G_2(\omega) \tag{3.43}$$

and (**frequency convolution**)

$$g_1(t) g_2(t) \Longleftrightarrow \frac{1}{2\pi} G_1(\omega) * G_2(\omega) \tag{3.44}$$

Proof: By definition,

$$\begin{aligned} \mathcal{F}|g_1(t) * g_2(t)| &= \int_{-\infty}^{\infty} e^{-j\omega t} \left[\int_{-\infty}^{\infty} g_1(\tau) g_2(t-\tau) d\tau \right] dt \\ &= \int_{-\infty}^{\infty} g_1(\tau) \left[\int_{-\infty}^{\infty} e^{-j\omega t} g_2(t-\tau) dt \right] d\tau \end{aligned}$$



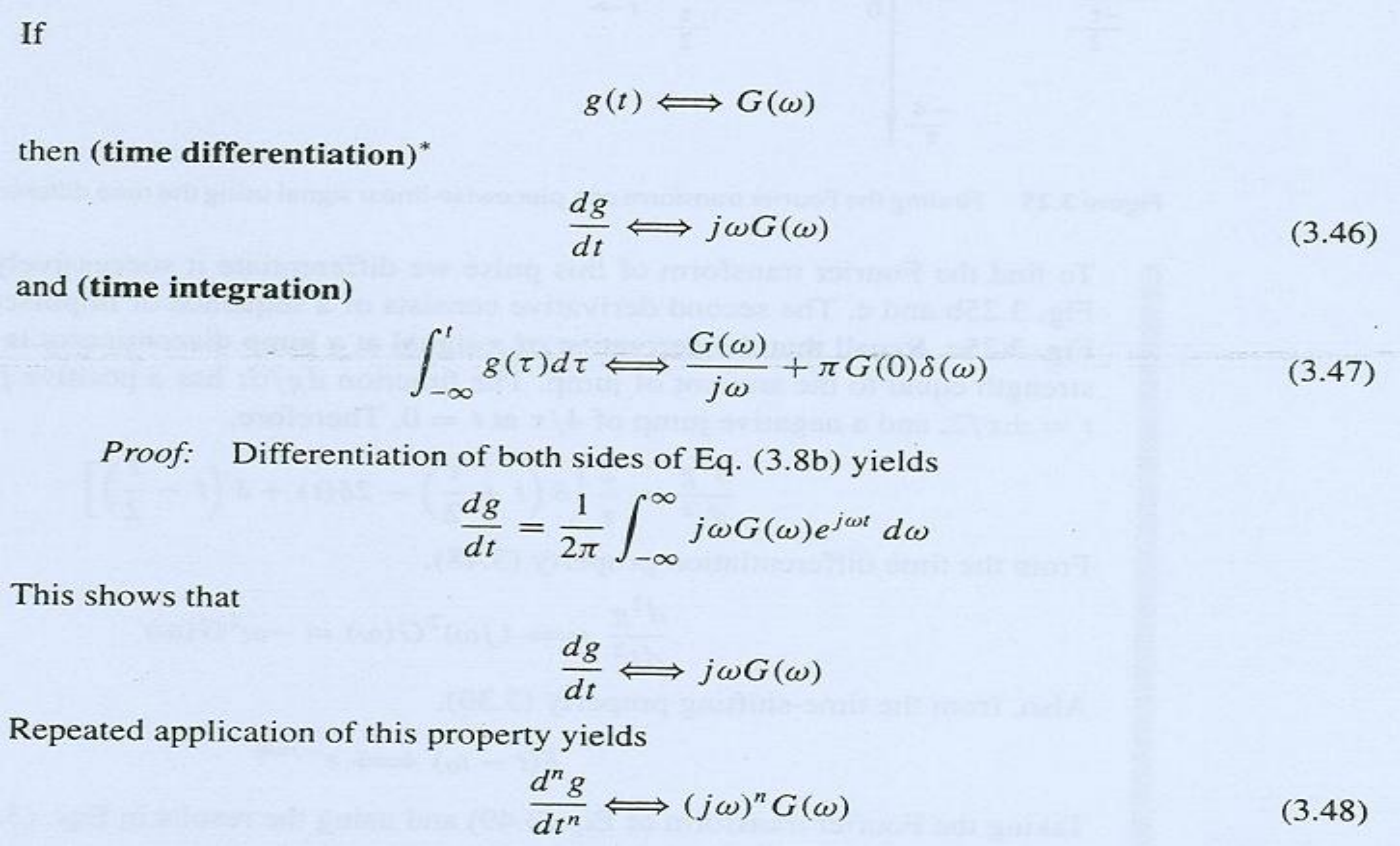
The inner integral is the Fourier transform of $g_2(t-\tau)$, given by [time-shifting property in Eq. (3.30)] $G_2(\omega)e^{-j\omega\tau}$. Hence,

$$\mathcal{F}[g_1(t) * g_2(t)] = \int_{-\infty}^{\infty} g_1(\tau)e^{-j\omega\tau}G_2(\omega)\,d\tau$$

$$= G_2(\omega)\int_{-\infty}^{\infty} g_1(\tau)e^{-j\omega\tau}\,d\tau = G_1(\omega)G_2(\omega)$$

The frequency convolution property (3.44) can be proved in exactly the same way by reversing the roles of $g(t)$ and $G(\omega)$.

Time Differentiation and Time Integration



If

$$g(t) \Longleftrightarrow G(\omega)$$

then **(time differentiation)***

$$\frac{dg}{dt} \Longleftrightarrow j\omega G(\omega) \tag{3.46}$$

and **(time integration)**

$$\int_{-\infty}^{t} g(\tau)d\tau \Longleftrightarrow \frac{G(\omega)}{j\omega} + \pi G(0)\delta(\omega) \tag{3.47}$$

Proof: Differentiation of both sides of Eq. (3.8b) yields

$$\frac{dg}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega G(\omega)e^{j\omega t}\, d\omega$$

This shows that

$$\frac{dg}{dt} \Longleftrightarrow j\omega G(\omega)$$

Repeated application of this property yields

$$\frac{d^n g}{dt^n} \Longleftrightarrow (j\omega)^n G(\omega) \tag{3.48}$$

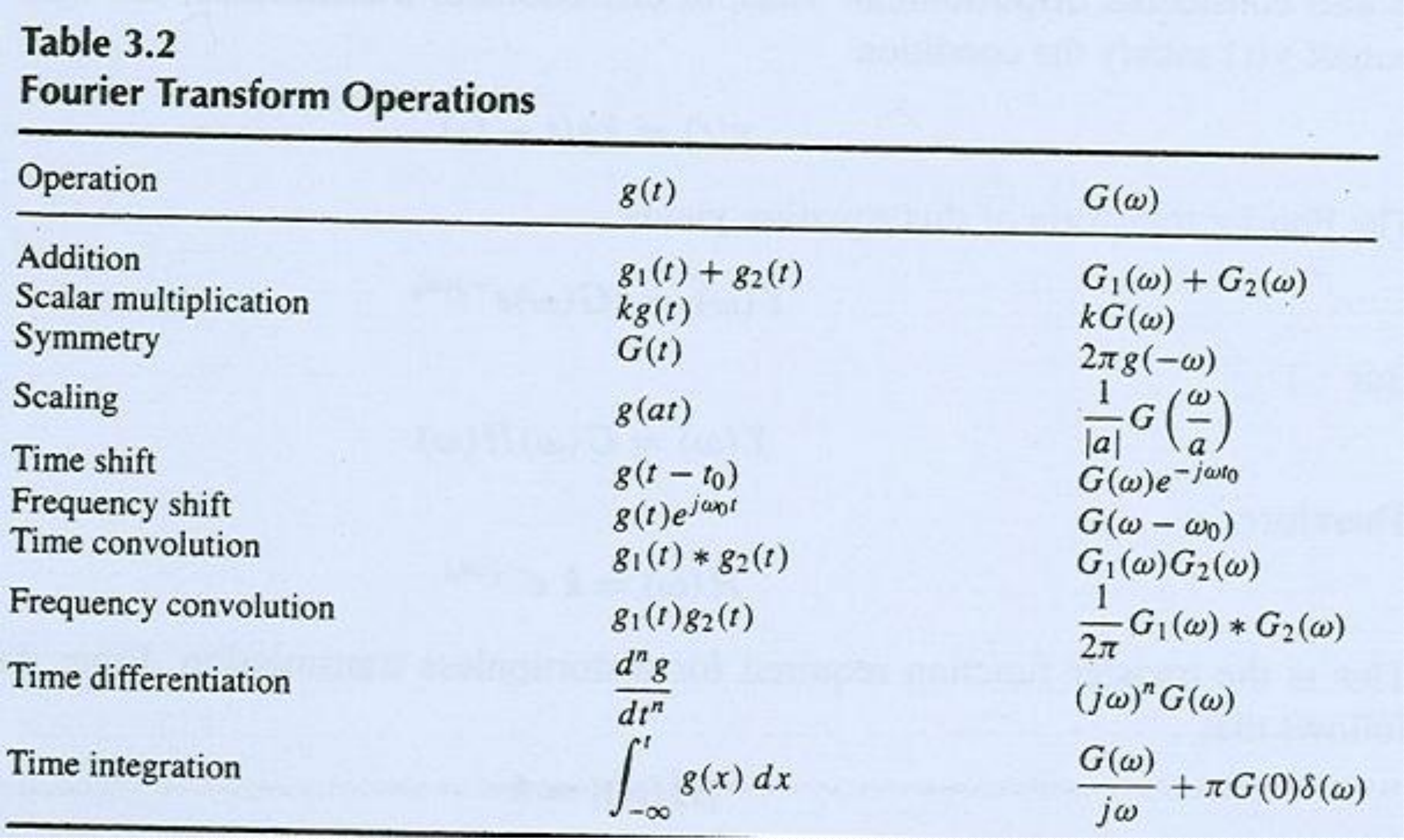


Table 3.2
Fourier Transform Operations

Operation	$g(t)$	$G(\omega)$
Addition	$g_1(t) + g_2(t)$	$G_1(\omega) + G_2(\omega)$
Scalar multiplication	$kg(t)$	$kG(\omega)$
Symmetry	$G(t)$	$2\pi g(-\omega)$
Scaling	$g(at)$	$\frac{1}{\lvert a\rvert}G\left(\frac{\omega}{a}\right)$
Time shift	$g(t - t_0)$	$G(\omega)e^{-j\omega t_0}$
Frequency shift	$g(t)e^{j\omega_0 t}$	$G(\omega - \omega_0)$
Time convolution	$g_1(t) * g_2(t)$	$G_1(\omega)G_2(\omega)$
Frequency convolution	$g_1(t)g_2(t)$	$\frac{1}{2\pi}G_1(\omega) * G_2(\omega)$
Time differentiation	$\frac{d^n g}{dt^n}$	$(j\omega)^n G(\omega)$
Time integration	$\int_{-\infty}^{t} g(x)\,dx$	$\frac{G(\omega)}{j\omega} + \pi G(0)\delta(\omega)$